Jensen and Popoviciu

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February 2016

1 Introduction

Hi, dear reader! In the forthcoming pages we intend to introduce Jensen's inequality and a stringer version of it, namely, Popoviciu's Inequality. We then look at some examples. We would like to conclude after that with a few problems for the reader. Note that some basic introduction to differential calculus is assumed. Anyway, there's a remark to make: Jensen is an incredibly powerful weapon so make sure you add it to your arsenal!

Note: Most of the problems have been taken from [1], which is indeed an excellent book on inequalities, and is pretty exhaustive in its content.

2 Jensen's Inequality: Introduction

We start with a definition of convex and concave functions:

Definiton 1. A function defined on the interval (a, b) is said to *convex* (respectively *concave*) iff for any $x \in (a, b)$ we have $f"(x) \geq 0$ (respectively ≤ 0). Further, the function is said to be *strictly convex* (respectively *strictly concave*) if equality does not hold.

Jensen's Inequality. Let $f : (a, b) \to R$ be a convex function on the interval (a, b). Further let $e_1, e_2, .., e_n$ be positive real numbers satisfying $e_1 + e_2 + ... + e_n = 1$. Then for all $x_1, ..., x_n \in (a, b)$ we have

$$f(\sum_{i=1}^{n} e_i x_i) \leq \sum_{i=1}^{n} e_i f(x_i).$$

Exercise. Prove Jensen's inequality using induction or any other suitable method.

Example 1. Consider the function $f(x) = x^2$. Then $f"(x) = 2 > 0$. Hence by

Jensen we obtain

$$f(\frac{\sum_{i=1}^{n} x_i y_i}{x_1+...x_n}) \leq \frac{\sum_{i=1}^{n} x_i f(y_i)}{x_1+...x_n} \iff (x_1y_1+x_2y_2+...+x_ny_n)^2 \leq (x_1+...x_n)(x_1y_1^2+...+x_ny_n^2)$$

Take $x_i = b_i^2$ and $y_i = \frac{a_i}{b_i}$ to get

$$(a_1b_1+...+a_nb_n)^2 \leq (a_1^2+...+a_n^2)(b_1^2+...+b_n^2)$$

which is the well known Cauchy-Schwarz inequality.

Example 2. Now let us look at the function $f(x) = \frac{1}{x}$ on the interval R^+. Then $f"(x) = \frac{2}{x^3} > 0$. Thus we have

$$\frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \leq \frac{\sum_{i=1}^{n} m_i f(x_i)}{\sum_{i=1}^{n} m_i}$$

If we introduce $a_i = \frac{1}{x_i}$ then the above inequality tranlates into

$$\frac{m_1a_1+...+m_na_n}{m_1+...+m_n} \geq \frac{m_1a_1+...+m_na_n}{\frac{m_1}{a_1}+...+\frac{m_n}{a_n}}$$

which is just the weighted AM-HM inequality.

Similarly, on using Jensen on e^x with a suitable substitution, it is possible to obtain the weighted AM-GM inequality.

3 Popoviciu's Inequality : Introduction

As you have seen (and will see in the example section) Jensen is a pretty powerful tool. However, in some cases it is just too weak. For that we come up with another inequality, called *Popoviciu's Inequality*, which is stronger than Jensen.

Popoviciu's Inequality. Let $f : (x,y) \to R$ be a convex function on (x,y). Then for $a,b,c \in (x,y)$ we have

$$f(a)+f(b)+f(c)+3f(\tfrac{a+b+c}{3}) \geq 2f(\tfrac{b+c}{2})+2f(\tfrac{c+a}{2})+2f(\tfrac{a+b}{2}).$$

As you may have guessed, there exists a generalization, namely the *Weighted Popoviciu's inequality* for $w_1, w_2, w_3 \in R_0^+$ such that the sum of no two of the $w_i's$ is 0, we have

$$w_1f(a)+w_2f(b)+w_3f(c)+3f(\tfrac{w_1a+w_2b+w_3c}{w_1+w_2+w_3}) \geq$$
$$2f(\tfrac{w_2b+w_3c}{w_2+w_3})+2f(\tfrac{w_3c+w_1a}{w_3+w_1})+2f(\tfrac{w_1a+w_2b}{w_1+w_2}).$$

There also exists a generalization to more that three variables, but we do not consider that here. We would just like to remark that there exist various generalizations of this inequality, mainly by Vasile Cırtoaje, Yufei Zhao and Zhaobin. But here we confine ourselves to just the usual Popoviciu. Further, there exists an inequality, *Karmata's inequality* that is *very* useful in IMO problems, but not so much before the TST, so I have decided to not include it. Interested readers may look it up from various sources available online.

4 Example Problems

So let us start with a relatively simple example:

1. In a (possibly degenerate) triangle $\triangle ABC$, show that the following inequality holds:

$$sinA + sinB + sinC \leq \frac{3\sqrt{3}}{4}$$

and determine all cases of equality.

Solution. We note that $sinx$ is strictly concave on $[0, \pi]$ so the required inequality is a consequence of Jensen's inequality. Equality holds iff $\triangle ABC$ is equilateral.

2. (Zuming Feng) Let x, y, z be positive reals such that $x + y + z = xyz$. Prove that

$$\frac{1}{1+xy} + \frac{1}{1+yz} + \frac{1}{1+zx} \leq \frac{3}{4}$$

Solution. It is natural to note that the above is not anything else but $\sum_{cyc} \frac{x}{S+x} \leq \frac{3}{4}$ where $S = x + y + z = xyz$.Now consider $f(x) = \frac{x}{S+x}$. Then $f'(x) = \frac{S+x-S-2x}{(S+x)^2} = \frac{-x}{(S+x)^2} \implies f"(x) = \frac{-(S+x)^2+2x(S+x)}{(S+x)^4} = \frac{-(y+z)}{(S+x)^3} < 0$. So Jensen kills this problem.

Next I think we should pay a visit to the IMO problems.

2. (IMO 2001) Let a, b, c be positive real numbers. Prove that

$$\sum_{cyc} \frac{a}{\sqrt{a^2+8bc}} \geq 1$$

Solution. Before I start, I would like to mention (because my peers have written two excellent articles on Isolated Fudging and Cauchy) that there exist alternate solutions using Holder or Cauchy or even Isolated fudging. But since here we wish to solve it using Jensen, here's the solution.

So we first observe that the inequality is *homogeneous in a,b,c* (this is a very important tactic, and deserves an article of its own) so we impose the condition $a + b + c = 1$. Now we apply Jensen to $f(x) = \frac{1}{\sqrt{x}}$ with necessary substitutions to yield

$$\sum_{cyc} \frac{a}{\sqrt{a^2+8bc}} \geq \frac{1}{\sqrt{a^3+b^3+c^3+24abc}}$$

and we have $1 = (a+b+c)^3 = a^3+b^3+c^3+6abc+3(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b) \geq a^3+b^3+c^3+24abc$ by AM-GM so we get 7/7.

Now look at this unused problem from Romania 2005:

3. (Romania 2005, Unused) Let $a+b+c=1$ where a,b,c are positive reals. Then prove that

$$\frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{c+a}} + \frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3}{2}}$$

Solution. On seeing this problem, I was like "Really? Not afraid of giving away free points?". This is just Jensen applied to $\frac{1}{\sqrt{x}}$ (this $\frac{1}{\sqrt{x}}$ seems to turn up very often in Olympiad Problems!) along with $ab+bc+ca \leq \frac{(a+b+c)^2}{3} = \frac{1}{3}$.

The function $f(x) = \frac{1}{\sqrt{x}}$ is very useful! Look at this:

4. (Serbia, Montenegro 2005) For positive reals a,b,c prove the inequality

$$\sum_{cyc} \frac{a}{\sqrt{b+c}} \geq \sqrt{\frac{3}{2}(a+b+c)}.$$

Solution. Homogeneous inequality + Jensen.

Now lets go for this problem from USAMO 2003.

5. (USAMO 2003) For positive reals a,b,c prove the inequality

$$\sum_{cyc} \frac{(2a+b+c)^2}{2a^2+(b+c)^2} \leq 8.$$

Solution. Homogeneous so let $a+b+c=1$, and thus we need to prove that $\sum_{cyc} \frac{a^2+2a+1}{3a^2-2a+1} \leq 8$. Set $f(x) = \frac{(x+1)^2}{2x^2+(1-x)^2}$, find $f"(x) = \frac{12(4x^3+3x^2-6x+1)}{(3x^2-2x+1)^3} \geq 0$ (prove this (AM-GM) ! :P). Now use Jensen: the rest is uninteresting and is left to the reader.

Unfortunately, due to my passport job, I couldn't gather examples of Popoviciu except some which I have left as exercises. I hope that you have understood the helpfulness of Jensen and in a similar way, Popoviciu. Now try a few problems yourself.

5 Problems

These problems are **not** arranged in any order of difficulty.

1. (Junior Balkan 2002) For $a,b,c > 0$ prove the inequality

$$\sum_{cyc} \frac{1}{b(a+b)} \geq \frac{27}{2(a+b+c)^2}.$$

2. (IMO 1995) Let $a, b, c \in R^+$ such that $abc = 1$. Prove the inequality

$$\sum_{cyc} \frac{1}{a^3(b+c)} \geq \frac{3}{2}.$$

3. (Vietnam 1998) Let $x_1, x_2, ..., x_n$ be positive reals such that

$$\frac{1}{x_1+1998} + \frac{1}{x_2+1998} + ... + \frac{1}{x_n+1998} = \frac{1}{1998}.$$

Prove that

$$\frac{\sqrt[n]{x_1 x_2 ... x_n}}{n-1} \geq 1998.$$

4. (Mildorf) Let a, b, c be the sides of a triangle. Prove that

$$\sum_{cyc} \frac{a}{\sqrt{2b^2+2c^2-a^2}} \geq \sqrt{3}.$$

5. (MOP 2002) Let a, b, c be positive reals. Prove the inequality

$$\sum_{cyc} \left(\frac{2a}{b+c}\right)^{\frac{2}{3}} \geq 3.$$

6. (USAMO 1998) Let $a_0, a_1, ..., a_n$ be numbers from the interval $(0, \pi/2)$ such that

$$\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) + \cdots + \tan(a_n - \frac{\pi}{4}) \geq n - 1.$$

Prove that

$$\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}.$$

7. (MR 6, 2006) For any triangle $\triangle ABC$, prove that the following inequality holds:

$$\sum_{cyc} \cos(\frac{A}{2}) \cot(\frac{A}{2}) \geq \frac{\sqrt{3}}{2}(\cot(\frac{A}{2}) + \cot(\frac{B}{2}) + \cot(\frac{C}{2})).$$

8. (India 1995, Kiran Kedlaya) Let $x_1, ..., x_n$ be positive reals summing to 1. Prove that

$$\sum_{cyc} \frac{x_1}{\sqrt{1-x_1}} \geq \sqrt{\frac{n}{n-1}}.$$

9. (Po-Shen Loh) Let $m \geq n$ be positive integers. Prove that every graph with m edges and n vertices has at least $\frac{m^2}{n}$ "V-shapes", which are defined to be unordered triples of vertices which have exactly to vertices between themselves.

6 References

[1] *Inequalities*, Zdravko Cvetkovski

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